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SOME HISTORICAL REMARKS ON NUMBER THEORY

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One of the main catalysts in the historical evolution of the theory of numbers and indeed of mathematics generally has been, as Professor Dieudonné has already remarked, the study of certain Diophantine equations, that is equations with integer variables. The most notorious of these is perhaps Fermat's equation

$x^n + y^n = z^n$ which, as is well-known, has been the subject of a great volume of research by amateur and professional mathematicians alike and which, moreover, through the theory of ideals and unique factorization can be regarded as the source of our modern theory of algebraic numbers. I should like to speak today, however, about another equation, of almost equal fame, which happens to have been solved in principle during the past few weeks. It provides, I think, a good illustration of the fusion of ideas about which Professor Dieudonné has just been speaking, and it will also enable me to describe some of the principal developments in number theory that have taken place in the last hundred years.

E.C. Catalan conjectured in 1844 that the only solution of the equation $x^m - y^n = 1$ in integers x, y, m, n is that given by $3^2 - 2^3 = 1$. The first results in this context, however, go back long before this date. Indeed, in the middle ages, Levi ben Gerson (also called Leo Hebraeus) showed that there is certainly no other solution with $x = 3, y = 2$, and Euler proved in 1738 that the same holds with $m = 2, n = 3$. Furthermore, a variety of special cases were solved by Frénicle de Bessy, V.A. Lebesgue and many others. In recent times W.J. LeVeque proved that given x, y there is at most one solution in integers m, n ; and J.W.S. Cassels, by refining LeVeque's work, showed that in fact, if m, n are primes, then they must divide y, x respectively. Cassels' work involved an old technique of C.D.T. Runge concerning the expansion of algebraic functions, but was otherwise essentially elementary. From an entirely different direction, C.L. Siegel proved in 1929 that, given m, n there are only finitely many x, y ; and recently I obtained an effective version of Siegel's Theorem, thereby establishing an explicit upper bound for x, y in terms of m, n . Both the work of Siegel and myself rested heavily on the theory of Diophantine approximation, but whereas Siegel's arguments depended on his refinements of Thue's results together with the Mordell-Weil Theorem, mine, on the other hand, utilized the most recent developments in transcendence theory. Perhaps I may now digress a little and sketch the history of the latter.

The subject was in fact initiated by Liouville in 1844, the

same year as that in which Catalan raised his conjecture; and it would probably have come as a considerable surprise to Liouville to learn that his simple observation to the effect that numbers of the form $\sum 10^{-n!}$ can satisfy no algebraic equation with integer coefficients, would one day lead to a solution of Catalan's problem. The first substantial contribution to the subject, however, was made by Hermite in 1873 when he proved that e , the natural base for logarithms, is transcendental. As Professor Dieudonné has already pointed out, Hermite's success rested on his skillful use of function theory, employed in this instance to generate simultaneous approximations to certain exponential series, but Hermite was not the first to recognize the importance of function theory in the investigation of arithmetical questions. Indeed, in his famous memoir of 1860, Riemann introduced the Zeta-function, ζn^{-s} , where the variable s is complex, and it was studies in this connection that led some thirty years later to the celebrated proof of the prime-number theorem by Hadamard and de la Vallée Poussin. Hermite's work was generalized by Lindemann in 1882, and various simplifications were later introduced by Weierstrass and others. These arguments furnished, in particular, the transcendence of the logarithmic function for algebraic values of the variable and so, more especially, the transcendence of π and a solution to the ancient Greek problem of the quadrature of the circle.

In 1909 Thue obtained the first general result on the finiteness of the number of solutions of a class of Diophantine equations by way of his improvement on Liouville's original result of 1844. The advance was achieved by constructing polynomials in two unknowns possessing zeros to a high order, rather than polynomials in one unknown as employed by Liouville. Later work of Roth and others has involved polynomials in many variables. Thue's work, as well as Hermite's, was also much extended by Siegel, who obtained, in particular, results on the transcendence of the Bessel functions and other hypergeometric series. And, in recent times, Thue's ideas have been employed by Stepanov and Schmidt to yield an elementary proof of the Riemann hypothesis for curves, much simpler than that originally given by Weil.

The next major progress was made by Gelfond in 1929 when, by extending the extrapolation techniques that he had previously employed in connection with studies on integral integer-valued functions, he succeeded in proving the transcendence of e^π ; and this work led a few years later to the Gelfond-Schneider solution to Hilbert's seventh problem. In 1966 I obtained a generalization of the Gelfond-Schneider theorem relating to linear forms in the logarithms of algebraic numbers, and refinements of this result have found a variety of applications to the effective resolution of Diophantine problems. One of the main features of the work is the utilization of functions of several complex

variables, rather than functions of a single variable as employed by Gelfond, and the functions have become successively more complicated as research in this field has continued.

Now I should like to return briefly to Catalan's equation. It was observed recently by a young Dutch mathematician, R. Tijdeman (*Acta Arithmetica*, to appear), that if m, n are odd primes, as one can assume, then, by elementary factorization, one has $x = kX^n + 1$, $y = \ell Y^m - 1$ for some integers X, Y , where k is 1 or m^{n-1} and ℓ is 1 or n^{m-1} . Indeed these expressions were utilized in the paper of Cassels referred to earlier, and they probably go back at least as far as Euler. Furthermore, if one substitutes for x, y , in the original Catalan equation and then takes logarithms, one sees at once that a particular linear form in three logarithms of rational numbers must, in a certain sense, be small. But now, as Tijdeman ingeniously shows, recent results in transcendence theory to which I have just referred, yield absolute bounds for m and n , and, from another result mentioned earlier, also derived from the theory of linear forms in logarithms, the latter then furnish bounds for x and y . Hence, in principle, Catalan's conjecture can be checked by a finite amount of computation.

Well, what does this tell us about the historical evolution of mathematics? First it is clear that a very important role has been played by a few key problems, centers of attraction, in Professor Dieudonné's terminology. This may be more true of number theory than other branches of mathematics but I believe that all good work has been guided to some extent by such centers. The general trend of the particular field that I have been discussing is difficult to summarize, since it has involved in its development many novel twists and turns; but one obvious element in the evolution has been the successful blending, or fusion, of ideas from number theory and algebra with the progressively wider use of classical function theory. And it is this convergence of diverse concepts that forms the essential ingredient, I believe, in the creation of an active theory. According to Professor Dieudonné, the study of transcendental numbers is only just on its way to becoming a "method". Given, however, the diverse nature of the problems which it has been instrumental in solving, there seems little doubt that it reached the latter stage several years ago, and it would appear, in fact that it is already on the path of becoming, in Professor Dieudonné's language, a center of radiation. For more information regarding the details of the subject I refer to my forthcoming book entitled *Transcendental Number Theory*, published by Cambridge University Press.

Finally I have been asked to mention some other unsolved problems in number theory, less well-known perhaps than Catalan's conjecture, but which may be stated equally simply. Of course, it is easy to write down unsolved problems but, generally speaking, number theorists are interested in them only if their solutions

appear to lie near the periphery of their skills, and this severely limits the range. I shall, however, mention two outstanding questions, the solution to either of which would probably lead to considerable advances. The first is the problem of Mordell as to whether the equation $x^3 + y^3 + z^3 = 3$ has any solutions in integers x, y, z other than those given by $(1, 1, 1)$ and $(4, 4, -5)$, and the second is the fundamental question of verifying that $e + \pi$ is irrational, or, more generally, that e and π are algebraically independent.

DISCUSSION

After hearing Baker's sequel to Dieudonné's talk, Morris Kline suggested that the word "fusion" had been used in two different senses. In one sense, it can mean bringing several fields together to create a new field; in another, it can mean drawing methods together in a new manner, while preserving the sovereignty of the original fields in other places.

Putnam then commented: "I think that when one says that mathematics or science develops in general by fusion, one should add that such fusion is often only possible after a new idea has been thought of which is orthogonal to earlier ways of thinking. Take set theory for example. People are beginning for the first time to use techniques from general mathematics, measure theory, topology, etc., in set theory. But that became possible only after Paul Cohen introduced the method of 'forcing'."

Dieudonné added that every example he gave was the work of only one or two persons; it is highly individual work. The introduction of a new idea immediately changes the whole scene. There is no comparison of algebraic geometry before and after Riemann; it is almost a completely different science.

Birkhoff recapitulated that an abstract morphology of the development of knowledge leaves out this essential point, that a new idea invented by a creative individual is as important or more important than anything else.

The following question was then posed by Professor Novikoff: "I know in number theory there are many instances of isolated problems that seem somewhat eccentric at first. They call for the best efforts of excellent people, and after a long time have a surprising history of successful results. Others remain unsolved and continue, I presume, to call forth such efforts. Are there any problems which a number theorist would conceive as "bad" ideas or are we only hearing about the successes?"

Dieudonné supplied odd perfect numbers as an example upon which no "sensible" work has been done. Novikoff then asked more specifically, "Do you finally decide that it is non-successful enough to be non-interesting or does it become more interesting because it is older and still unsolved?" Dieudonné replied that

no one has as yet come up with the original idea needed to solve the problem. Birkhoff cautioned that it was very dangerous for a group of mathematicians to gather together and decide what are the "right" problems or methods. He propounded the need for independent individuals to work in unconventional and unfashionable ways.

Dieudonné emphasized the importance of knowing whether or not a particular problem has a *finite* number of solutions. "As I said, until the Mordell-Weil or Thue Theorem was proved, results were still somewhat restricted to homogenous polynomials. The first general result was that any homogenous polynomial of degree greater than three has only a finite number of solutions. This was superseded by the Siegel Theorem, which made it true for any equation of genus one with integral coefficients. The best one can hope for, say for Fermat's Theorem, is for someone to prove that there is only a finite number of solutions. To prove no solution at all seems to be hopeless."

The importance of finding particular solutions of Catalan's equation, which had been presented by Alan Baker, versus proving that only a finite number of solutions exist was debated.

Kahane shifted the discussion back to Novikoff's original questions, by asking if he wanted to know why this type of equation was interesting. Novikoff replied in the affirmative, adding that it was not clear to him from the discussion why any particular equation would be interesting or not. Certainly, he would like to know why Catalan's equation is of concern to number theorists.

One possible answer was provided by Kahane, who said that it was interesting for mathematicians to study that which was simple to state, yet difficult to solve. Dieudonné stated that these were given results, whereas others had previously given no results at all. Novikoff then conjectured that this was why mathematicians could be so confident that they would never run out of problems.

Finally, Kahane commented that Baker's talk was a "masterpiece of what Baker as a mathematician can do on the history of mathematics."